

# Frictionless Random Dynamics: Hydrodynamical Formalism

Radosław Czopnik

Institute of Theoretical Physics, University of Wrocław,  
PL - 50 209 Wrocław, Poland

and

Piotr Garbaczewski\*

Institute of Physics, University of Zielona Góra,  
PL - 65 069 Zielona Góra, Poland

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## Abstract

We investigate an undamped random phase-space dynamics in deterministic external force fields (conservative and magnetic ones). By employing the hydrodynamical formalism for those stochastic processes we analyze microscopic kinetic-type "collision invariants" and their relationship to local conservation laws (moment equations) in the fully nonequilibrium context. We address an issue of the continual heat absorption (particles "energization") in the course of the process and its possible physical implementations.

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## 1 Introduction

The major topic discussed in the present paper will be the frictionless random motion which is introduced in close affinity with second-order processes driven by white noise, [1, 2]. In the course of motion an unrestricted "energization" of particles is possible, a phenomenon which has not received much attention in the literature, although appears to be set on convincing phenomenological grounds in specific (nontypical) physical surroundings, [3, 4].

A classic problem of an irreversible behaviour of a particle embedded in the large encompassing system, [5, 6, 7] pertains to the random evolution of tracer particles in a fluid (like e.g. those performing Brownian motion). In that case, a particle - bath coupling is expected to imply quick relaxation towards a stationary probability distribution (equilibrium Maxwellian for a given a priori temperature  $T$  of the surrounding fluid/bath) and thus making the system amenable to standard fluctuation-dissipation theorems. The dissipation of randomly acquired energy *back to* the bath is here enforced by dynamical friction mechanisms, [8, 9, 10, 7], although *no* balance between stochastic forcing (random

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\*E-mail: pgar@proton.if.uz.zgora.pl

acceleration effects) and dissipative losses of energy is in fact in existence in the course of the process, unless asymptotically, [6, 11].

The traditional picture of a tracer particle in a fluid/gas, even if modeled in terms of stochastic diffusion-type processes, is usually considered in direct reference to the Boltzmann kinetic theory. That justifies an exploitation of properly defined microscopic collision rules to yield dynamical friction (c.f. [7, 12]) and then to allow for a "derivation" (which breaks down the microscopic conservation laws on the way, [13]) of the Brownian dynamics in suitable scaling limits. The crucial assumption about a rapid decay of correlation functions amounts to strong friction and short relaxation time, [7, 9, 8].

Let us mention that, as reveals the hydrodynamical formalism of the Brownian motion, [14], Brownian particles "while being thermalized", show up a continual net absorption of heat/energy from the reservoir (that point is normally disregarded in view of the very short relaxation times): so-called kinetic temperature grows up to the actual temperature of the bath, in parallel with the mean kinetic energy of the Brownian particle, [11, 15, 6], c.f. also [16, 17]. Indeed, [17] chap.2, for  $t > 0$  we encounter on the average a continual "heating" phenomenon in the course of which an asymptotic equilibrium value  $kT/m$  is achieved:  $E|\vec{u}|^2 = \frac{q}{\beta}[1 - \exp(-2\beta t)] \rightarrow \frac{kT}{m}$  where  $q = \beta kT/m$ , while  $\vec{u}$  stands for velocity random variable. In this case the bath is thought of as insensitive to this process and is supposed to remain in the perpetual state of "statistical rest".

(Notice that such situation is surely in conflict with dynamical approaches to the thermalization of the Brownian particle, [18, 13], which take into account that the particle-thermostat coupling acts both ways (consider reaction forces, third Newton law in the mean etc.). In consequence, it is *both* the Brownian particle *and* the dissipating agent (bath, thermostat) which need to thermalize together i.e. simultaneously evolve towards a common state of statistical equilibrium, cf. [18].)

In below we shall consider purely classical prototype stochastic models which depart from most traditional Brownian motion in its Langevin (white noise), Kramers and Smoluchowski versions. However our focus will be on an extremally nonequilibrium situation when no thermalisation is possible. Then, stationary (equilibrium) state is definitely *out* of the reach, no fluctuation-dissipation relationships are in existence and the untamed heating phenomenon is the most conspicuous characteristic of the randomly forced dynamics.

## 1.1 Phenomenological motivations

In contrast to the well established frictional "Brownian motion standards", we shall explore the *opposite* regime of dissipation - free theory. The departure from the standard theory appears to be quite radical, since it refers to a different physics, where dissipative time scales (if any) are *much longer* than the time duration of processes of interest, including the particle life-time.

We are particularly motivated by two papers [3, 4] devoted to the thermal energization of particles

which is due to impulsive stochastic forcing. In fact, frictionless stochastic processes were invoked to analyze situations present in magnetospheric environments. Specifically, one deals there with charged particles in a locally uniform magnetic field which experience intermittent stochastic electrical forcing.

However, then a serious conceptual obstacle is present, if the theory is considered without suitable reservations: one needs to be able to control and eventually tame the kinetic energy growth, since in the absence of friction we deal with an efficient particle "energization" mechanism. That involves as well an important issue (not addressed in the present paper) of suitable thermalization/thermostatting mechanisms that would possibly tame the otherwise unlimited "energization" of particles or make it effectively irrelevant on the time-scales of interest.

Small random impulses occur within ambient fields in such way that the smooth deterministic dynamics of a particle is irregularly disrupted by sudden and very short bursts of low-intensity noise. Without friction, we deal with an undamped random walk and no fluctuation-dissipation relationship can be established.

Phenomena of that kind are seldom mentioned in the traditional statistical physics research, since they may be realized in rather unusual, from the point of view of the common practice, environments.

Indeed, [20] the physical nature of environments in which the random propagation is investigated, appears to be crucial. In the original papers [3, 4], the environment of interest was an astrophysical or space plasma, where densities and therefore dissipation are often reduced by many orders of magnitude. That amounts to dissipation time scales which are much longer than those on which any random forcing may take place. Moreover, physical processes associated with the thermalization were eliminated in view of the fact that particles entered geometrical regions of space where the plasma instabilities (producing stochastic electric fields fluctuations) and thus the source of randomness had disappeared. In contrast to typical laboratory environments (e.g. plasmas), the high spatial heterogeneity present in space plasmas makes random forcing to occur only within very limited regions of the non-stationary environment and *not* to occur everywhere and for all times.

That means that it is the very nature of the environment which causes the source of energization to disappear and reappear spontaneously, while typical random bath models heavily rely on the noise spatial homogeneity and its "eternal" existence, [8].

## 1.2 Hydrodynamical formalism for random dynamics

To elucidate the formal roots of our strategy and reasons for studying the frictionless random dynamics by means of the hydrodynamical formalism, let us call our attention back to the standard Ornstein-Uhlenbeck (dissipative) framework. The Langevin equation for mass  $m$  particle in an external field of force (we consider a conservative case)  $\vec{F} = \vec{F}(\vec{x}) = -\vec{\nabla}V$  reads:

$$\frac{d\vec{x}}{dt} = \vec{u} \quad (1)$$

$$\frac{d\vec{u}}{dt} = -\beta\vec{u} + \frac{\vec{F}}{m} + \vec{A}(t) \quad (2)$$

Random acceleration  $\vec{A}(t)$  obeys the white noise statistics:  $\langle A_i(s) \rangle = 0$  and  $\langle A_i(s) A_j(s') \rangle = 2q\delta(s - s')\delta_{ij}$ , where  $i = 1, 2, 3$ .

Since things are specialized to the *standard* Brownian motion, we know a priori that noise intensity is determined by a parameter  $q = D\beta^2$  where  $D = \frac{kT}{m\beta}$ , while the friction parameter  $\beta$  is given by the Stokes formula  $m\beta = 6\pi\eta a$  (or its analogue in case of the Lorentz gas, [7]). Consequently, the effect of the surrounding medium on the motion of the particle is described by two parameters: friction constant  $\beta$  and bath temperature  $T$ . Assumptions about the asymptotic (equilibrium) Maxwell-Boltzmann distribution and the fluid reaction upon the moving particle are here implicit, [9].

The resulting (Markov) phase - space diffusion process is completely determined by the transition probability density  $P(\vec{x}, \vec{u}, t | \vec{x}_0, \vec{u}_0, t_0)$ , which is typically expected to be a fundamental solution of the Kramers equation:

$$\frac{\partial P}{\partial t} + \vec{u} \cdot \nabla_{\vec{x}} P + \left[ -\beta\vec{u} + \frac{\vec{F}}{m} \right] \cdot \nabla_{\vec{u}} P = q \nabla_{\vec{u}}^2 P \quad (3)$$

The associated spatial Smoluchowski diffusion process with a forward drift  $\vec{b}(\vec{x}) = \frac{\vec{F}}{m\beta}$  is analyzed in terms of increments of the normalized Wiener process  $\vec{W}(t)$ . The infinitesimal increment of the configuration (position) random variable  $\vec{X}(t)$  reads:

$$d\vec{X}(t) = \frac{\vec{F}}{m\beta} dt + \sqrt{2D} d\vec{W}(t) \quad (4)$$

The related Fokker-Planck equation for the spatial probability density  $\rho(\vec{x}, t)$  reads  $\partial_t \rho = D\Delta\rho - \vec{\nabla} \cdot (\rho \vec{b})$  and explicitly employs the large friction regime, [10, 9, 13]. In fact, we take for granted that both time and space scales *of interest* (i.e. those upon which the accumulation of relevant random events prove to be significant) largely exceed the relaxation time interval  $\beta^{-1}$  and that dominant contributions "of interest" come from velocities  $|\vec{u}| \leq (kT/m) = (q/\beta)^{1/2}$  and that the corresponding variation of  $\vec{r}$  is sufficiently small (actually it is of the order  $|\vec{u}|/\beta \equiv (q/\beta^3)^{1/2}$ ), [9].

Under those assumptions the Fokker -Planck equation for the spatial Markov process arises as the scaling ( $\beta \gg 1$ ) limit of the 0-th order moment equation associated with the original Kramers law of random phase-space dynamics. In fact, by following the traditional pattern of hydrodynamical formalism, [14, 16], we infer the closed system of two (which is special to Markovian diffusions !) local

conservation laws for the Smoluchowski process, [11, 13]:

$$\partial_t \rho + \vec{\nabla} \cdot (\vec{v} \rho) = 0 \quad (5)$$

$$(\partial_t + \vec{v} \cdot \vec{\nabla}) \vec{v} = \vec{\nabla} (\Omega - Q). \quad (6)$$

Here (we use a short-hand notation  $\vec{v}(\vec{x}, t) \doteq \vec{v}$ )

$$\vec{v}(\vec{x}, t) = \frac{\vec{F}}{m\beta} - D \frac{\vec{\nabla} \rho}{\rho} \quad (7)$$

defines so-called current velocity of Brownian particles and, when inserted to Eq. (5), transforms the continuity equation into the Fokker-Planck equation, [10].

Eq. (6) stands for the scaling limit of the first order moment equation derivable from the kinetic equation (3) and directly corresponds to the familiar Euler equation, characterizing the momentum conservation law in the lowest order approximation of kinetic theory based on the Boltzmann equation, [16].

However, the large friction regime enforces here a marked difference in the local momentum conservation law, in comparison with the standard Euler equation for a nonviscous fluid or gas. Namely, instead of the kinetic theory motivated expression for e.g. rarified gas:

$$(\partial_t + \vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{\vec{F}}{m} - \frac{\vec{\nabla} P}{\rho} \quad (8)$$

where  $P(\vec{x})$  stands for the pressure function (to be fixed by a suitable equation of state) and  $\vec{F}$  is the very same (conservative  $-\vec{\nabla} V$ ) force acting upon particles as that appearing in the Kramers equation (3), the Smoluchowski regime (6) employs the emergent volume force (notice the positive sign)  $+\vec{\nabla} \Omega$  instead of  $-\vec{\nabla} V$ :

$$\Omega = \frac{1}{2} \left( \frac{\vec{F}}{m\beta} \right)^2 + D \vec{\nabla} \cdot \left( \frac{\vec{F}}{m\beta} \right) \quad (9)$$

and the pressure-type contribution  $-\vec{\nabla} Q$  where, [13] (see also [21, 22])

$$Q = 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \quad (10)$$

and  $\Delta = \vec{\nabla}^2$  is the Laplace operator, does not leave any room for additional constraints upon the system (like e.g. the familiar equation of state).

To have a glimpse of a dramatic difference between physical messages conveyed respectively by equations (6) and (8), it is enough to insert in (8) the standard equation of state  $P(\vec{x}) = \alpha \rho^\beta$  with  $\alpha, \beta > 0$  and choose  $\vec{F} = -\omega^2 \vec{x}$  to represent the harmonic attraction in Eqs. (2) - (9), see also [13].

Markovian diffusion processes with the inverted sign of  $\vec{\nabla}(\Omega - Q)$  in the local momentum conservation law (6) i. e. respecting

$$(\partial_t + \vec{v} \cdot \vec{\nabla}) \vec{v} = \vec{\nabla} (Q - \Omega) \quad (11)$$

instead of Eq. (6), were considered in Ref. [13] as implementations of the "third Newton law in the mean". Nonetheless, also under those premises, the volume force term  $-\vec{\nabla}\Omega$  in Eq. (11) does *not* in general coincide with the externally acting conservative force contribution (e.g. acceleration)  $\frac{1}{m}\vec{F} = -\frac{1}{m}\vec{\nabla}V$  akin to Eq. (8).

Accordingly, the effects of external force fields acting upon particles are significantly distorted while passing to the local conservation laws in the large friction (Smoluchowski) regime.

That becomes even more conspicuous in case of the Brownian motion of a charged particle in the constant magnetic field, [15, 8, 23]. In the Smoluchowski (large friction) regime, friction completely smoothes out any rotational (due to the Lorentz force) features of the process. In the corresponding local momentum conservation law there is *no* volume force contribution at all and merely the "pressure-type" potential  $Q$  appears in a rescaled form, [15]:

$$Q = \frac{\beta^2}{\beta^2 + \omega_c^2} \cdot 2D^2 \frac{\Delta \rho^{1/2}}{\rho^{1/2}} \quad (12)$$

where  $\beta$  is the (large) friction parameter and  $\omega_c = \frac{q_e B}{mc}$  is the cyclotron frequency of the charge  $q_e$  particle in a constant homogeneous magnetic field  $\vec{B} = (0, 0, B)$ . Clearly, for moderate frequency values  $\omega_c$  (hence the magnetic field intensity) and sufficiently large  $\beta$  even this minor scaling remnant of the original Lorentz force would effectively disappear, yielding Eq. (10).

This observation is to be compared with results of Refs. [3, 4] where, in the absence of friction, the rotational Lorentz force input clearly survives when passing to the local conservation laws, in plain contrast with the Smoluchowski regime. By disregarding friction (or considering it to be irrelevant on suitable time scales) it is possible to reproduce exactly the conservative external force acting upon particles in the local conservation laws, [24].

Thus the low noise regime, if combined with the sufficiently short duration time of processes of interest (much lower than the dissipation time scales) sets the range of validity of local conservation laws in the classic Euler form (8), when demanded to comply with the stochastic diffusion-type microscopic dynamics assumption.

## 2 Random motion of a free particle

### 2.1 Résumé

Let us consider the Langevin-type equation for a particle which is not suffering any friction while being subject to random accelerations

$$\begin{aligned}\frac{dx}{dt} &= u \\ \frac{du}{dt} &= A(t)\end{aligned}\tag{13}$$

We fix initial conditions:  $x(t_0) = x_0$ ,  $u(t_0) = u_0$ ,  $t_0 = 0$ . The fluctuating term  $A(t)$  is assumed to display a standard white noise statistic: (i)  $A(t)$  is independent of  $u$ , (ii)  $\langle A(s) \rangle = 0$  and  $\langle A(s) A(s') \rangle = 2q\delta(s - s')$ , where  $q$  is here left as an unspecified (albeit a priori physical) parameter.

This random dynamics induces a transition probability density  $P(x, u, t | x_0, u_0, t_0)$  which uniquely defines the corresponding phase-space Markovian diffusion process executed by  $(x, u)$ . Here, the function  $P(x, u, t | x_0, u_0, t_0)$  is the fundamental solution of the Fokker-Planck equation associated with the above Langevin equation:

$$\frac{\partial P}{\partial t} = -u \frac{\partial P}{\partial x} + q \frac{\partial^2 P}{\partial u^2}.\tag{14}$$

A fundamental solution of Eq. (14) was first given by Kolmogorov [25] and is a literature classic, [1, 2]. For the uniformity of further argumentation, we shall here follow a direct solution method [26, 17] which employs properties of the involved Wiener process  $W_t$ .

Since  $\sqrt{2q}W_t = \int_0^t A(s) ds$ , the distribution of the random variable  $S = u - u_0$  is gaussian with mean zero and variance  $2qt$ . We define the displacement of the particle as  $x - x_0 = \int_0^t u(\eta) d\eta$ . Hence, we have  $R = x - x_0 - u_0 t = \sqrt{2q} \int_0^t W_s ds$ .

Because  $S$  and  $R$  are gaussian random variables, their joint probability distribution  $W(R, S)$  is also gaussian and is given in terms of mean values and covariances of  $S$  and  $R$ . The mean values are equal to zero, the covariance matrix is given that  $C = \begin{pmatrix} \text{cov}(R, R) & \text{cov}(R, S) \\ \text{cov}(S, R) & \text{cov}(S, S) \end{pmatrix} = 2q \begin{pmatrix} \frac{1}{3}t^3 & \frac{1}{2}t^2 \\ \frac{1}{2}t^2 & t \end{pmatrix}$  and  $\text{cov}(W_s, W_t) = s \wedge t$ .

$$\text{Thus, } W(R, S) = \frac{1}{2\pi} \left( \frac{1}{\det C} \right)^{\frac{1}{2}} \exp \left[ -\frac{1}{2} \begin{pmatrix} R & S \end{pmatrix} C^{-1} \begin{pmatrix} R \\ S \end{pmatrix} \right]$$

Since the problem of finding the transition probability distribution  $P(x, u, t | x_0, u_0, t_0)$  is equivalent to finding the joint probability distribution  $W(R, S)$  of random variables  $S$  and  $R$ , we have thus arrived at the following expression for (time homogeneous)  $P(x, u, t | x_0, u_0, t_0)$ :

$$P(x, u, t | x_0, u_0, t_0 = 0) = \frac{1}{2\pi} \frac{\sqrt{12}}{2qt^2} \exp \left[ -\frac{(u - u_0)^2}{4qt} - \frac{3 \left( x - x_0 - \frac{u + u_0}{2} t \right)^2}{qt^3} \right].\tag{15}$$

We encounter a situation when standard fluctuation-dissipation relationships are manifestly invalid. Nonetheless, it is useful to mention a useful computational connection with the familiar damped case.

Namely, let us consider the Langevin equation for the free (would be Brownian) particle with the friction term

$$\frac{dx}{dt} = u \quad (16)$$

$$\frac{dv}{dt} = -\beta u + A(t) \quad (17)$$

but no a priori presumed (fluctuation-dissipation) relationship between  $\beta$  and  $q$ . We again fix the initial conditions:  $x(t_0) = x_0$ ,  $u(t_0) = u_0$ ,  $t_0 = 0$ .

Except for considering  $\beta$  and  $q$  as independent parameters, we can directly exploit the classic result [9] for the corresponding transition probability density  $P(x, u, t | x_0, u_0, t_0 = 0)$  solving the Kramers equation (compare e.g. Eq. (3))

$$\frac{\partial P}{\partial t} + u \frac{\partial P}{\partial x} = \beta \frac{\partial}{\partial u} (Pu) + q \frac{\partial^2 P}{\partial u^2} \quad (18)$$

Its solution is known to arise in the form of a joint probability density  $W(R, S)$  of random variables  $S = u - u_0 e^{-\beta t}$  and  $R = x - x_0 - \beta^{-1} (1 - e^{-\beta t}) u_0$

$$W(R, S) = \left( \frac{1}{4\pi^2 (fg - h^2)} \right)^{\frac{1}{2}} \exp \left[ -\frac{gR^2 - 2hRS + fS^2}{2(fg - h^2)} \right] \quad (19)$$

where the auxiliary coefficients are given that  $f = \frac{q}{\beta^3} (2\beta t - e^{-2\beta t} + 4e^{-\beta t} - 3)$ ,  $g = \frac{q}{\beta} (1 - e^{-2\beta t})$ ,  $h = \frac{q}{\beta^2} (1 - e^{-\beta t})^2$ .

Since we assume that the intensity of noise  $q$  is *independent* of the friction parameter  $\beta$ , we can safely take the limit of  $\beta \rightarrow 0$  in the joint probability density  $W(R, S)$ . Then, in view of the obvious limiting values for  $S = v - v_0$ ,  $R = x - x_0 - v_0 t$ ,  $f = \frac{2}{3} q t^3$ ,  $g = 2qt$ ,  $h = qt^2$  we readily arrive at the previous frictionless-case expression.

Let us stress that it is the assumption about the independence of parameters  $q$  and  $\beta$  which takes us away from the standard theory of Brownian motion. We cannot any longer expect that thermal equilibrium conditions can be approached by standard dissipation mechanisms. We also recall that in case of the "normal" dissipative free Brownian theory [9], its physical formulation is based on the requirement that the distribution of velocities must be given by the stationary Maxwell-Boltzmann probability density with variance  $\frac{kT}{m}$  as  $t \rightarrow \infty$ . Moreover, the traditional Einstein-Smoluchowski theory relates the spatial diffusion coefficient  $D$  to  $\beta$  by the fluctuation - dissipation relation  $D = \frac{kT}{m\beta}$ . That appears to be consistent with the Kramers picture of the phase-space random dynamics only if we a priori assume that  $q = \frac{kT}{m} \beta$ , [9, 10]. Then  $q = D\beta^2$  and large  $\beta$  definitely implies large  $q$ .

## 2.2 Kramers equation and local conservation laws

The stationary (time homogeneous) transition probability density for the diffusion process without friction in phase-space has the form (15) and is the fundamental solution to Kramers (Fokker-Planck



type) equation (14). We are interested in passing to a hydrodynamical picture, following the traditional recipes [14, 16]. To that end we need to propagate certain initial probability density and investigate effects of the random dynamics. Let us choose most obvious example of:

$$\rho_0(x, u) = \left(\frac{1}{2\pi a^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini})^2}{2a^2}\right) \left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2b^2}\right). \quad (20)$$

so that at time  $t$  we have  $\rho(x, u, t) = \int P(x, u, t | x_0, u_0, t_0 = 0) \rho_0(x_0, u_0) dx_0 du_0$ .

Since  $P(x, u, t | x_0, u_0, t_0)$  is the fundamental solution of the Kramers equation, the joint density  $\rho(x, u, t)$  is also the solution and can be written in the familiar, [9], form of Eq. (19) for  $\rho(x, u, t) = W(R, S)$ . However, in the present case functional entries are adopted to the frictionless motion and read as follows:  $S = u - u_{ini}$ ,  $R = x - x_{ini} - u_{ini}t$  with

$$\begin{aligned} f &= a^2 + b^2 t^2 + \frac{2}{3} q t^3 \\ g &= b^2 + 2qt \\ h &= b^2 t + q t^2. \end{aligned} \quad (21)$$

The marginals  $\rho(x, t) = \int \rho(x, u, t) du$  and  $\rho(u, t) = \int \rho(x, u, t) dx$  are

$$\rho(x, t) = \left(\frac{1}{2\pi f}\right)^{\frac{1}{2}} \exp\left(-\frac{R^2}{2f}\right) = \left(\frac{1}{2\pi(a^2 + b^2 t^2 + \frac{2}{3} q t^3)}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini} - u_{ini}t)^2}{2(a^2 + b^2 t^2 + \frac{2}{3} q t^3)}\right) \quad (22)$$

and

$$\rho(u, t) = \left(\frac{1}{2\pi g}\right)^{\frac{1}{2}} \exp\left(-\frac{S^2}{2g}\right) = \left(\frac{1}{2\pi(b^2 + 2qt)}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2(b^2 + 2qt)}\right) \quad (23)$$

Let us introduce an auxiliary (reduced) distribution  $\widetilde{W}(S|R) = \frac{W(S,R)}{\int W(S,R) dS}$  where in the denominator we recognize the marginal spatial distribution  $\int W(S, R) dS \doteq w$  of  $W(S, R)$ .

Following the standard hydrodynamical picture method [14, 16, 11] we define local (configuration space conditioned) moments:  $\langle u \rangle_x = \int u \widetilde{W} du$  and  $\langle u^2 \rangle_x = \int u^2 \widetilde{W} du$ . First, we compute

$$\langle u \rangle_x = u_{ini} + \frac{h}{f} R = u_{ini} + \frac{b^2 t + q t^2}{a^2 + b^2 t^2 + \frac{2}{3} q t^3} [x - x_{ini} - u_{ini}t] \quad (24)$$

$$\langle u^2 \rangle_x - \langle u \rangle_x^2 = \left(g - \frac{h^2}{f}\right) = \frac{q t^3 (2 b^2 + q t) + 3 a^2 (b^2 + 2 q t)}{3 a^2 + t^2 (3 b^2 + 2 q t)} \quad (25)$$

Now, the first two moment equations for the Kramers equation are easily derivable. Namely, the continuity (0-th moment) and the momentum conservation (first moment) equations come out in the form

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (\langle u \rangle_x w) = 0 \quad (26)$$

$$\frac{\partial}{\partial t} (\langle u \rangle_x w) + \frac{\partial}{\partial x} (\langle u^2 \rangle_x w) = 0 \quad (27)$$

These equations yield the local momentum conservation law in the familiar form

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \frac{\partial}{\partial x} \right) \langle u \rangle_x = -\frac{1}{w} \frac{\partial P_{kin}}{\partial x} \quad (28)$$

where we encounter the standard [16] textbook notion of the pressure function

$$P_{kin}(x, t) = \left[ \langle u^2 \rangle_x - \langle u \rangle_x^2 \right] w(x, t). \quad (29)$$

The marginal density  $w$  obeys  $\frac{\nabla w}{w} = -2f\nabla \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right]$  and that in turn implies

$$-\frac{1}{w} \frac{\partial P_{kin}}{\partial x} = 2(fg - h^2) \nabla \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right]. \quad (30)$$

As a consequence, the local conservation law (28) takes the form

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \nabla \right) \langle u \rangle_x = -\frac{\nabla P_{kin}}{w} = +2(fg - h^2) \nabla \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] \doteq +\nabla Q \quad (31)$$

where (we point out the plus sign in the above, see e.g. Eq. (11))

$$fg - h^2 = a^2 b^2 + 2a^2 qt + \frac{2}{3} b^2 qt^3 + \frac{1}{3} q^2 t^4 \doteq D^2(t) \quad (32)$$

and by adopting the notation  $D^2(t) \doteq fg - h^2$  we get  $-\frac{1}{w} \frac{\partial P_{kin}}{\partial x} = +\nabla Q$  with the functional form of  $Q$  given by Eq. 10. Here, instead of a diffusion constant  $D$  we insert the (positive) time -dependent function  $D(t)$ .

With those notational adjustments, we recognize in Eq. (31) a consistent Euler form of the local momentum conservation law, in case of vanishing volume forces (c.f. Eq. (8) and compare that with Eq. (11) again).

It is useful to mention that by employing Eq.(23) we can readily evaluate the mean kinetic energy associated with the frictionless free dynamics. Namely, there holds

$$\langle u(t)^2 \rangle = \langle u(t) \rangle^2 = b^2 + 2qt \quad (33)$$

where  $\langle u(t) \rangle = \langle u(0) \rangle = u_{ini}$ . That is to be compared with the standard Brownian motion expression, [17]:  $\langle u(t)^2 \rangle = \frac{q}{\beta} [1 - \exp(-2\beta t)]$ . In the present case we definitely encounter an untamed, continual heating (energization of Refs. [3, 4]) mechanism.

Let us observe that limits  $a \rightarrow 0, b \rightarrow 0$  are here under control and allow to reproduce a consistent hydrodynamical formalism for the transition probability density of the frictionless process. Indeed, in

that case, we readily reproduce observations of Refs. [1, 2] that  $\langle u^2(t) \rangle = 2qt$ , while for the spatial mean displacement we would get  $\langle x^2(t) \rangle = \frac{2}{3}qt^3$ . The spatial process shows features typical of an anomalous transport and obeys the Fokker-Planck equation  $\partial_t \rho = (qt^2)\Delta \rho$ .

### 3 Charged particle in a constant magnetic field

Our analysis of a free particle subject to random accelerations and no friction can be extended to the case when external deterministic force fields are in action. As a specific example we shall discuss the dynamics of a charged particle in a constant magnetic field, see e.g. also [15, 23, 29, 30].

Magnetic field effect upon the charged particle in practically frictionless random environments was the main objective of Refs. [3, 4] in the formulation of a kinematic mechanism for particle energization. Those authors were primarily interested in the application of their model in concrete physical situations, hence they did not concentrate on finding the general solution to the involved probabilistic problem. They have evaluated the mean square velocity and gave computer assisted visualization of the dynamics. However no general descriptions of the related velocity space nor phase-space stochastic process were given. Thus no information was available about spatial features of the dynamics, nor about the associated macroscopic (hydrodynamical formalism) balance equations.

#### 3.1 Velocity-space process

For a charged particle in a constant magnetic field and fluctuating electric environment, we consider the equation of motion in the form

$$\frac{d\vec{u}}{dt} = \frac{q_e}{mc} \vec{u} \times \vec{B} + \vec{A}(t) \quad (34)$$

where  $\vec{u}$  denotes the velocity of the particle of an electric charge  $q_e$  and mass  $m$ . A fluctuating part  $\vec{A}(t)$  represents random forces of electric origin which are to obey the standard white noise statistics assumptions (c.f. Section 2.1).

Let us assume for simplicity that magnetic field  $\vec{B}$  is directed along the z-axis of a Cartesian reference frame so that  $\vec{B} = (0, 0, B)$  and  $B = \text{const}$ . In this case, Eq. (34) takes the form

$$\frac{d\vec{u}}{dt} = -\Lambda \vec{u} + \vec{A}(t) \quad (35)$$

where

$$\Lambda = \begin{pmatrix} 0 & -\omega_c & 0 \\ \omega_c & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and  $\omega_c = \frac{q_e B}{mc}$  denotes the cyclotron frequency.

The corresponding transition probability density  $P(\vec{u}, t | \vec{u}_0, t_0 = 0)$  of the time-homogeneous Markov process with the initial condition:  $P(\vec{u}, t | \vec{u}_0) \rightarrow \delta^3(\vec{u} - \vec{u}_0)$  as  $t \rightarrow 0$  has the form:

$$P(\vec{u}, t | \vec{u}_0) = \left( \frac{1}{4\pi qt} \right)^{\frac{3}{2}} \exp \left( -\frac{|\vec{u} - U(t) \vec{u}_0|^2}{4qt} \right) \quad (36)$$

where  $U(t) = \exp(-\Lambda t)$ , see e.g. Ref. [15].

$P(\vec{u}, t | \vec{u}_0, t_0 = 0)$  is the solution of the Fokker-Planck equation:

$$\frac{\partial P}{\partial t} = -\omega_c [\nabla_{\vec{u}} \times P \vec{u}]_{i=3} + q \nabla_{\vec{u}}^2 P \quad (37)$$

where  $[\nabla_{\vec{u}} \times P \vec{u}]_{i=3} = \frac{\partial}{\partial u_1} (P u_2) - \frac{\partial}{\partial u_2} (P u_1)$ . The expression (36) can be readily extended to the case  $t_0 = s \neq 0, t \geq s$ ,

$$P(\vec{u}, t | \vec{v}, s) = \left( \frac{1}{4\pi q(t-s)} \right)^{\frac{3}{2}} \exp \left( -\frac{|\vec{u} - U(t-s) \vec{v}|^2}{4q(t-s)} \right). \quad (38)$$

Let us consider in some detail the motion of an individual particle which has an initial velocity  $\vec{v}_0$  at  $t_0 = 0$ , and the initial probability density  $\varrho(\vec{v}, s = 0) = \varrho_0(\vec{v}) = \delta(\vec{v} - \vec{v}_0)$ . The propagation in time is controlled by the transition density distribution  $P(\vec{u}, t | \vec{v}, s)$  according to

$$\varrho(\vec{u}, t) = \int P(\vec{u}, t | \vec{v}, s = 0) \varrho_0(\vec{v}) d\vec{v} \quad (39)$$

$$\varrho(\vec{u}, t) = \left( \frac{1}{4\pi qt} \right)^{\frac{3}{2}} \exp \left( -\frac{|\vec{u} - U(t) \vec{u}_0|^2}{4qt} \right). \quad (40)$$

By evaluating mean values of the resultant density for  $i = 1, 2$  velocity components, we obtain:

$$\langle u_i(t) \rangle = \int_{-\infty}^{\infty} u_i \varrho(\vec{u}, t) d\vec{u} \quad (41)$$

$$\langle \vec{u}(t) \rangle = (\langle u_1(t) \rangle, \langle u_2(t) \rangle) = U(t) \vec{u}_0 \quad (42)$$

$$\langle u_i^2(t) \rangle = \int_{-\infty}^{\infty} u_i^2 \varrho(\vec{u}, t) d\vec{u} \quad (43)$$

$$\sigma_i^2 = \langle u_i^2(t) \rangle - \langle u_i(t) \rangle^2 = 2qt \quad (44)$$

we realize that the kinetic energy of the particle increases linearly with time:  $\langle u^2(t) \rangle - \langle u^2(0) \rangle = 4qt$ . This result obviously coincides with less explicit calculations of [3].

If we define the displacement  $\vec{r}$  of the particle as  $\vec{r} - \vec{r}_0 = \int_0^t \vec{u}(\eta) d\eta$ , we can extend the description of the process to the phase space. An attempt to specify the transition probability density  $P(\vec{r}, \vec{u}, t | \vec{r}_0, \vec{u}_0, t_0 = 0)$  is equivalent to finding the joint distribution  $W(\vec{S}, \vec{R})$  of random vectors  $\vec{S}$  and  $\vec{R}$ , which are defined by the equations

$$\vec{S} = \vec{u} - U(t) \vec{u}_0 \quad (45)$$

$$\vec{R} = \vec{r} - \vec{r}_0 - \Lambda^{-1} [1 - U(t)] \vec{u}_0 \quad (46)$$

where  $\Lambda^{-1} = \begin{pmatrix} 0 & \omega_c^{-1} \\ -\omega_c^{-1} & 0 \end{pmatrix}$  and  $U(t) = \begin{pmatrix} \cos \omega_c t & \sin \omega_c t \\ -\sin \omega_c t & \cos \omega_c t \end{pmatrix}$ .

The joint probability density of  $\vec{S}$  and  $\vec{R}$  has the form

$$W(\vec{S}, \vec{R}) = \frac{1}{4\pi^2 (fg - h^2 - k^2)} \exp \left( -\frac{f|\vec{S}|^2 + g|\vec{R}|^2 - 2h\vec{S} \cdot \vec{R} + 2k(\vec{S} \times \vec{R})_{i=3}}{2(fg - h^2 - k^2)} \right) \quad (47)$$

with the auxiliary coefficients:

$$g = 2qt$$

$$f = 4q \frac{1}{\omega_c^2} \left( t - \frac{1}{\omega_c} \sin \omega_c t \right) \quad (48)$$

$$h = \frac{2q}{\omega_c^2} (1 - \cos \omega_c t)$$

$$k = -\frac{2q}{\omega_c} \left[ t - \frac{1}{\omega_c} \sin \omega_c t \right].$$

Eq. (47) is the solution to the Fokker-Planck equation in the phase space:

$$\frac{\partial W}{\partial t} + \vec{u} \nabla_{\vec{r}} W = -\omega_c [\nabla_{\vec{u}} \times W \vec{u}]_{i=3} + q \nabla_{\vec{u}}^2 W \quad (49)$$

where  $[\nabla_{\vec{u}} \times W \vec{u}]_{i=3} = \frac{\partial}{\partial u_1} (W u_2) - \frac{\partial}{\partial u_2} (W u_1)$ .

### 3.2 Kramers equation and local conservation laws

The formula (47) is an explicit solution to the Kramers Fokker-Planck equation and coincides with the phase-space transition probability density  $P(\vec{r}, \vec{u}, t | \vec{r}_0, \vec{u}_0, t_0 = 0)$ .

Let us consider the following initial phase-space probability density

$$\rho_0(\vec{r}_0, \vec{u}_0) = \frac{1}{2\pi a^2} \exp \left( -\frac{(\vec{r}_0 - \vec{r}_{ini})^2}{2a^2} \right) \frac{1}{2\pi b^2} \exp \left( -\frac{(\vec{u}_0 - \vec{u}_{ini})^2}{2b^2} \right) \quad (50)$$

which is propagated in the course of the stochastic process so that at time  $t$  we have  $\rho(\vec{r}, \vec{u}, t) = \int P(\vec{r}, \vec{u}, t | \vec{r}_0, \vec{u}_0, t_0 = 0) \rho_0(\vec{r}_0, \vec{u}_0) d\vec{r}_0 d\vec{u}_0$

Since  $P(\vec{r}, \vec{u}, t | \vec{r}_0, \vec{u}_0, t_0 = 0)$  is the fundamental solution of the Kramers equation the joint density  $\rho(\vec{r}, \vec{u}, t)$  is also the solution. This joint density can be put into the form

$$\rho(\vec{r}, \vec{u}, t) = W(\vec{S}, \vec{R}) = \frac{1}{4\pi^2 (fg - h^2 - k^2)} \exp \left( -\frac{f|\vec{S}|^2 + g|\vec{R}|^2 - 2h\vec{S} \cdot \vec{R} + 2k(\vec{S} \times \vec{R})_{i=3}}{2(fg - h^2 - k^2)} \right) \quad (51)$$

where

$$\vec{S} = \vec{u} - U(t) \vec{u}_{ini} \quad (52)$$

$$\vec{R} = \vec{r} - \vec{r}_{ini} - \Lambda^{-1} [1 - U(t)] \vec{u}_{ini} \quad (53)$$

$$g = b^2 + 2qt \quad (54)$$

$$f = \frac{2b^2\omega_c(1 - \cos\omega_ct) + a^2\omega_c^3 + 4q(\omega_ct - \sin\omega_ct)}{\omega_c^3} \quad (55)$$

$$h = \frac{2q(1 - \cos\omega_ct) + b^2\omega_c\sin\omega_ct}{\omega_c^2} \quad (56)$$

$$k = -\frac{2q(\omega_ct - \sin\omega_ct) + b^2\omega_c(1 - \cos\omega_ct)}{\omega_c^2}. \quad (57)$$

The marginals  $\rho(\vec{r}, t) = \int \rho(\vec{r}, \vec{u}, t) d\vec{u}$  and  $\rho(\vec{u}, t) = \int \rho(\vec{r}, \vec{u}, t) d\vec{r}$  read

$$\rho(\vec{r}, t) = \frac{1}{2\pi f} \exp\left(-\frac{|\vec{R}|^2}{2f}\right) = \quad (58)$$

$$\frac{1}{2\pi \frac{1}{\omega_c^3} (2b^2\omega_c(1 - \cos\omega_ct) + a^2\omega_c^3 + 4q(\omega_ct - \sin\omega_ct))} \times \exp\left(-\frac{|\vec{r} - \vec{r}_{ini} - \Lambda^{-1} [1 - U(t)] \vec{u}_{ini}|^2}{2 \frac{1}{\omega_c^3} (2b^2\omega_c(1 - \cos\omega_ct) + a^2\omega_c^3 + 4q(\omega_ct - \sin\omega_ct))}\right)$$

and

$$\rho(\vec{u}, t) = \frac{1}{2\pi g} \exp\left(-\frac{|\vec{S}|^2}{2g}\right) = \frac{1}{2\pi (b^2 + 2qt)} \exp\left(-\frac{|\vec{u} - U(t) \vec{u}_{ini}|^2}{2(b^2 + 2qt)}\right). \quad (59)$$

Notice that in view of  $\langle u_i^2(t) \rangle - \langle u_i(t) \rangle^2 = b^2 + 2qt$ , where  $\langle u_i(t) \rangle = [U(t) \vec{u}_{ini}]_i$  with  $i = 1, 2$ , and  $\langle \vec{u}(t) \rangle^2 = |U(t) \vec{u}_{ini}|^2 = |\vec{u}_{ini}|^2 = \langle \vec{u}(0) \rangle^2$ , we have the same as before (see e.g. Section 2) indicative of the linear in time energy growth:

$$\langle \vec{u}^2(t) \rangle - \langle \vec{u}(0) \rangle^2 = 2b^2 + 4qt. \quad (60)$$

We shall perform major steps of the hydrodynamical analysis following the pattern of Sec. 2. To this end we introduce the auxiliary distribution  $\widetilde{W}(\vec{S}|\vec{R}) = \frac{w(\vec{S}, \vec{R})}{\int w(\vec{S}, \vec{R}) d\vec{S}}$  which has an explicit form

$$\widetilde{W}(\vec{S}|\vec{R}) = \frac{1}{2\pi \frac{1}{f} (fg - h^2 - k^2)} \exp\left(-\frac{|\vec{S} - \vec{m}|^2}{2 \frac{1}{f} (fg - h^2 - k^2)}\right) \quad (61)$$

where  $\vec{m} = \frac{1}{f} (hR_1 - kR_2, hR_2 + kR_1)$ ,  $\vec{R}$  and  $\vec{S}$  are given by equations (52) and (53) respectively.

We define local moments as  $\langle u_i \rangle_{\vec{R}} = \int u_i \widetilde{W} d\vec{u}$  and  $\langle u_i^2 \rangle_{\vec{R}} = \int u_i^2 \widetilde{W} d\vec{u}$  where  $i = 1, 2$ . It is apparent from the Eq. (61) that  $\langle \vec{u} \rangle_{\vec{R}} = (\langle u_1 \rangle_{\vec{R}}, \langle u_2 \rangle_{\vec{R}}) = U(t) \vec{u}_{ini} + \vec{m}$  and  $\langle u_i^2 \rangle_{\vec{R}} = \langle u_i \rangle_{\vec{R}}^2 + \frac{1}{f} (fg - h^2 - k^2)$ .

The first two moment equations for the Kramers equation (49) are easily derivable:

$$\partial_t w + \vec{\nabla} [\langle \vec{u} \rangle_{\vec{R}} w] = 0 \quad (62)$$

$$\partial_t [\langle u_1 \rangle_{\vec{R}} w] + \frac{\partial}{\partial r_1} [\langle u_1^2 \rangle_{\vec{R}} w] + \frac{\partial}{\partial r_2} [\langle u_1 \rangle_{\vec{R}} \langle u_2 \rangle_{\vec{R}} w] = \omega_c \langle u_2 \rangle_{\vec{R}} w \quad (63)$$

$$\partial_t [\langle u_2 \rangle_{\vec{R}} w] + \frac{\partial}{\partial r_2} [\langle u_2^2 \rangle_{\vec{R}} w] + \frac{\partial}{\partial r_1} [\langle u_1 \rangle_{\vec{R}} \langle u_2 \rangle_{\vec{R}} w] = -\omega_c \langle u_1 \rangle_{\vec{R}} w \quad (64)$$

Those equations imply the local momentum conservation law

$$\begin{aligned} \left[ \partial_t + \langle \vec{u} \rangle_{\vec{R}} \cdot \vec{\nabla} \right] \langle \vec{u} \rangle_{\vec{R}} &= -\Lambda \langle \vec{u} \rangle_{\vec{R}} - \frac{1}{w} \vec{\nabla} \cdot \overleftrightarrow{P} \\ &= \frac{q_e}{mc} \langle \vec{u} \rangle_{\vec{R}} \times \vec{B} - \frac{1}{w} \vec{\nabla} \cdot \overleftrightarrow{P} \end{aligned} \quad (65)$$

where  $\vec{\nabla} \cdot \overleftrightarrow{P}$  is a vector which  $i$ -th component is equal to  $\sum_j \frac{\partial P_{ij}}{\partial r_j}$ ,  $i, j = 1, 2$  and  $\overleftrightarrow{P}$  denotes the tensor consisting of components  $P_{ij} = \langle (u_i - \langle u_i \rangle_{\vec{R}}) (u_j - \langle u_j \rangle_{\vec{R}}) \rangle_{\vec{R}} w$ , they are given that  $P_{11} = (\langle u_1^2 \rangle_{\vec{R}} - \langle u_1 \rangle_{\vec{R}}^2) w$ ,  $P_{22} = (\langle u_2^2 \rangle_{\vec{R}} - \langle u_2 \rangle_{\vec{R}}^2) w$  and  $P_{12} = P_{21} = 0$ .

Let us point out that

$$\langle u_1^2 \rangle_{\vec{R}} - \langle u_1 \rangle_{\vec{R}}^2 = \langle u_2^2 \rangle_{\vec{R}} - \langle u_2 \rangle_{\vec{R}}^2 = g - \frac{h^2 + k^2}{f} \quad (66)$$

displays a linear growth in time  $\sim qt$  for large  $t$ .

Owing to relation

$$\left( -\frac{1}{w} \frac{\partial P_{11}}{\partial x}, -\frac{1}{w} \frac{\partial P_{22}}{\partial y} \right) = 2(fg - h^2 - k^2) \vec{\nabla} \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] \quad (67)$$

the local momentum conservation law (65) takes the form

$$\left[ \partial_t + \langle \vec{u} \rangle_{\vec{R}} \cdot \vec{\nabla} \right] \langle \vec{u} \rangle_{\vec{R}} = \frac{q_e}{mc} \langle \vec{u} \rangle_{\vec{R}} \times \vec{B} + 2(fg - h^2 - k^2) \vec{\nabla} \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] \quad (68)$$

where the crucial time-dependent coefficient reads

$$fg - h^2 - k^2 = a^2 b^2 - \frac{8q^2}{\omega_c^4} + \frac{4b^2 qt}{\omega_c^2} + \frac{4q^2 t^2}{\omega_c^2} + 2a^2 qt + \frac{8q^2}{\omega_c^4} \cos(t\omega_c) - \frac{4b^2 q}{\omega_c^3} \sin(t\omega_c). \quad (69)$$

In contrast to the free dynamics coefficient (32), presently we need to maintain a proper balance between (small) noise intensity  $q$  and the (not too large) frequency  $\omega_c$  to guarantee a positivity of (69), thus allowing for a subsequent redefinition in terms of  $D^2(t)$ , c.f. Eqs. (32), (32) for comparison. If so, we can safely write (keep in mind that we consider exclusively the planar motion):

$$2(fg - h^2 - k^2) \vec{\nabla} \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] = + \vec{\nabla} Q \quad (70)$$

The important feature of this local conservation law is that on its right-hand side appears the very same force which is present in the Kramers equation (49), as opposed to the corresponding conservation law in the Smoluchowski diffusion regime, [13].

## 4 Harmonically bound particle

### 4.1 Velocity-space process

For completeness, let us analyze in detail the case of harmonically attracting force. We consider the one-dimensional harmonic oscillator with circular frequency  $\omega$ . The equation of motion reads (for further notational convenience we replace the previous  $A(t)$  by  $\xi(t)$ ):

$$\begin{aligned}\frac{dx}{dt} &= u \\ \frac{du}{dt} &= -\omega^2 x + \xi(t)\end{aligned}\tag{71}$$

or

$$\frac{d}{dt} \begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix}\tag{72}$$

and differs from the Langevin equation in the standard Ornstein-Uhlenbeck theory in that we have no friction term  $-\beta u$ .

The formal solution to this equation is given in terms of the resolvent operator  $R(t, t_0) = e^{A(t-t_0)}$ , where  $A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$  and has the form

$$\vec{x}(t) = R(t, t_0) \vec{x}_0 + \int_{t_0}^t R(t, s) \vec{\xi}(s) ds.\tag{73}$$

For the sake of simplicity of the formulae we introduce the following notation  $\vec{x}(t) = (x(t), u(t))$  and in our case  $\vec{\xi}(s) = (0, \xi(s))$ .

The characteristic equation of the matrix  $A$  is  $\lambda^2 + \omega^2 = 0$ , with the eigenvalues  $\lambda_1 = i\omega$  and  $\lambda_2 = -i\omega$ . The corresponding eigenvectors are  $x^1 = \begin{pmatrix} 1 \\ \lambda_1 \end{pmatrix}$  and  $x^2 = \begin{pmatrix} 1 \\ \lambda_2 \end{pmatrix}$ . In the base of eigenvectors  $\{x^1, x^2\}$  matrix  $A$  has the diagonal form  $A' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ .

Since  $e^{A'(t-t_0)} = \begin{pmatrix} e^{\lambda_1(t-t_0)} & 0 \\ 0 & e^{\lambda_2(t-t_0)} \end{pmatrix}$  we can now easily evaluate the resolvent operator  $R(t, t_0) = e^{A(t-t_0)} = B e^{A'(t-t_0)} B^{-1}$ , where  $B = \begin{pmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{pmatrix}$  is the matrix related to the change of base and  $B^{-1} = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{pmatrix}$ . Namely, we have:

$$R(t, t_0) = \frac{1}{\lambda_2 - \lambda_1} \begin{pmatrix} \lambda_2 e^{\lambda_1(t-t_0)} - \lambda_1 e^{\lambda_2(t-t_0)} & -e^{\lambda_1(t-t_0)} + e^{\lambda_2(t-t_0)} \\ \lambda_1 \lambda_2 e^{\lambda_1(t-t_0)} - \lambda_1 \lambda_2 e^{\lambda_2(t-t_0)} & -\lambda_1 e^{\lambda_1(t-t_0)} + \lambda_2 e^{\lambda_2(t-t_0)} \end{pmatrix}\tag{74}$$

which can be explicitly written as follows:

$$R(t, t_0) = \begin{pmatrix} \cos \omega(t-t_0) & \frac{1}{\omega} \sin \omega(t-t_0) \\ -\omega \sin \omega(t-t_0) & \cos \omega(t-t_0) \end{pmatrix}\tag{75}$$



The formal solution to (71) has the form

$$x = x_0 \cos \omega (t - t_0) + u_0 \frac{1}{\omega} \sin \omega (t - t_0) + \int_{t_0}^t \frac{1}{\omega} \sin \omega (t - s) \xi (s) ds \quad (76)$$

$$u = -x_0 \omega \sin \omega (t - t_0) + u_0 \cos \omega (t - t_0) + \int_{t_0}^t \cos \omega (t - s) \xi (s) ds. \quad (77)$$

Let us introduce the following notation

$$R = x - x_0 \cos \omega (t - t_0) - u_0 \frac{1}{\omega} \sin \omega (t - t_0) = \int_{t_0}^t \psi (s) \xi (s) ds \quad (78)$$

$$S = u + x_0 \omega \sin \omega (t - t_0) - u_0 \cos \omega (t - t_0) = \int_{t_0}^t \varphi (s) \xi (s) ds \quad (79)$$

where  $\psi (s) = \frac{1}{\omega} \sin \omega (t - s)$  and  $\varphi (s) = \cos \omega (t - s)$ .

If we define furthermore:

$$f = 2q \int_{t_0}^t \psi^2 (s) ds = \frac{2q}{\omega^2} \int_{t_0}^t \sin^2 \omega (t - s) ds = \frac{q}{\omega^2} (t - t_0) \left[ 1 - \frac{\sin 2\omega (t - t_0)}{2\omega (t - t_0)} \right] \quad (80)$$

$$g = 2q \int_{t_0}^t \varphi^2 (s) ds = 2q \int_{t_0}^t \cos^2 \omega (t - s) ds = q (t - t_0) \left[ 1 + \frac{\sin 2\omega (t - t_0)}{2\omega (t - t_0)} \right] \quad (81)$$

$$h = 2q \int_{t_0}^t \varphi (s) \psi (s) ds = \frac{2q}{\omega} \int_{t_0}^t \sin \omega (t - s) \cos \omega (t - s) ds = \frac{q}{\omega^2} \sin^2 \omega (t - t_0) \quad (82)$$

we end up with a function  $W (R, S)$  of the form (20) which can be read out as a transition probability density for the Markovian phase-space process (85), here denoted as  $P (x, v, t | x_0, v_0, t_0)$ , which clearly solves the appropriate Kramers-Fokker-Planck equation:

$$\frac{\partial P}{\partial t} + v \frac{\partial P}{\partial x} = \omega^2 x \frac{\partial P}{\partial v} + q \frac{\partial^2 P}{\partial v^2}. \quad (83)$$

## 4.2 Local conservation laws

As an initial density we choose  $\rho_0 (x, u)$  of Eq. (21) and propagate that according to  $\rho (x, v, t) = \int P (x, u, t | x_0, u_0, t_0 = 0) \rho_0 (x_0, v_0) dx_0 du_0$ .

Since  $P (x, u, t | x_0, u_0, t_0)$  is the fundamental solution of the Kramers equation, the joint density  $\rho (x, u, t)$  is also the solution. This joint density can be put into the form  $\rho (x, u, t) = W (R, S)$ , c.f. Eq. (96), where

$$S = u + x_{ini} \omega \sin \omega t - u_{ini} \cos \omega t \quad (84)$$

$$R = x - x_{ini} \cos \omega t - u_{ini} \frac{1}{\omega} \sin \omega t \quad (85)$$

$$f = \frac{\omega (b^2 + 2qt + a^2 \omega^2) + (- (b^2 \omega) + a^2 \omega^3) \cos(2t\omega) - q \sin(2t\omega)}{2\omega^3} \quad (86)$$

$$g = \frac{\omega (b^2 + 2qt + a^2 \omega^2) + \omega (b^2 - a^2 \omega^2) \cos(2t\omega) + q \sin(2t\omega)}{2\omega} \quad (87)$$

$$h = \frac{\sin(t\omega) (\omega (b^2 - a^2 \omega^2) \cos(t\omega) + q \sin(t\omega))}{\omega^2}. \quad (88)$$

The marginals  $\rho(x, t) = \int \rho(x, u, t) du$  and  $\rho(u, t) = \int \rho(x, u, t) dx$  read:

$$\begin{aligned} \rho(x, t) &= \left( \frac{1}{2\pi f} \right)^{\frac{1}{2}} \exp \left( -\frac{R^2}{2f} \right) = \\ &\left( \frac{1}{2\pi \frac{1}{2\omega^3} (\omega (b^2 + 2qt + a^2 w^2) + (- (b^2 \omega) + a^2 \omega^3) \cos(2t\omega) - q \sin(2t\omega))} \right)^{\frac{1}{2}} \\ &\times \exp \left( -\frac{(x - x_{ini} \cos \omega t - v_{ini} \frac{1}{\omega} \sin \omega t)^2}{2 \frac{1}{2\omega^3} (\omega (b^2 + 2qt + a^2 w^2) + (- (b^2 \omega) + a^2 \omega^3) \cos(2t\omega) - q \sin(2t\omega))} \right) \end{aligned}$$

and

$$\begin{aligned} \rho(u, t) &= \left( \frac{1}{2\pi g} \right)^{\frac{1}{2}} \exp \left( -\frac{S^2}{2g} \right) \\ &= \left( \frac{1}{2\pi \frac{1}{2\omega} (\omega (b^2 + 2qt + a^2 w^2) + \omega (b^2 - a^2 w^2) \cos(2t\omega) + q \sin(2t\omega))} \right)^{\frac{1}{2}} \\ &\times \exp \left( -\frac{(u + x_{ini} \omega \sin \omega t - u_{ini} \cos \omega t)^2}{2 \frac{1}{2\omega} (\omega (b^2 + 2qt + a^2 w^2) + \omega (b^2 - a^2 w^2) \cos(2t\omega) + q \sin(2t\omega))} \right) \end{aligned}$$

At this point we can evaluate  $\langle u(t) \rangle^2 = [x_{ini} \omega \sin(\omega t) - u_{ini} \cos(\omega t)]^2$  and next:

$$\langle u^2(t) \rangle - \langle u(t) \rangle^2 = \frac{\omega(b^2 + 2qt + a^2 w^2) + \omega(b^2 - a^2 w^2) \cos(2t\omega) + q \sin(2t\omega)}{2\omega} \quad (89)$$

to have clearly exemplified the kinetic energy growth (e.g.  $\langle u^2(t) \rangle - \langle u(0) \rangle^2$ ). Notice that by taking the  $\omega \rightarrow 0$  limit, we recover the force-free dynamics result (35).

As before, let us introduce the auxiliary distribution  $\widetilde{W}(S|R) = \frac{W(S,R)}{\int W(S,R) dS}$  which has the standard Gaussian form

$$\widetilde{W}(S|R) = \left( \frac{1}{2\pi \left( g - \frac{h^2}{f} \right)} \right)^{\frac{1}{2}} \exp \left( -\frac{\left| S - \frac{h}{f} R \right|^2}{2 \left( g - \frac{h^2}{f} \right)} \right) \quad (90)$$

We define local moments  $\langle u \rangle_x = \int u \widetilde{W} du$  and  $\langle u^2 \rangle_x = \int u^2 \widetilde{W} du$  so that

$$\langle u \rangle_x = -x_{ini} \omega \sin \omega t + u_{ini} \cos \omega t + \frac{h}{f} \left[ x - x_{ini} \cos \omega t - u_{ini} \frac{1}{\omega} \sin \omega t \right] \quad (91)$$

$$\langle u^2 \rangle_x = \langle u \rangle_x^2 + \left( g - \frac{h^2}{f} \right). \quad (92)$$

and local conservation laws follow:

$$\frac{\partial w}{\partial t} + \frac{\partial}{\partial x} (\langle u \rangle_x w) = 0 \quad (93)$$

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \frac{\partial}{\partial x} \right) \langle u \rangle_x = -\omega^2 x - \left[ \langle u^2 \rangle_x - \langle u \rangle_x^2 \right] \frac{1}{w} \frac{\partial w}{\partial x}. \quad (94)$$

The local momentum conservation law (94) takes the form

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \frac{\partial}{\partial x} \right) \langle u \rangle_x = -\omega^2 x + 2(fg - h^2) \nabla \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] \quad (95)$$

where the time-dependent coefficient reads:

$$fg - h^2 = \frac{-q^2 + 2b^2 q t \omega^2 + 2q^2 t^2 \omega^2 + 2a^2 b^2 \omega^4 + 2a^2 q t \omega^4 + q^2 \cos(2t\omega) + q\omega(-b^2 + a^2 \omega^2) \sin(2t\omega)}{2\omega^4}. \quad (96)$$

Again, a proper balance between the (small) intensity parameter  $q$  and the frequency  $\omega$  must be maintained to secure the positivity of (96), since then only the notation  $D^2(t)$  can be consistently introduced and the pressure-type contribution  $+\nabla Q$  explicitly identified in Eq. (95).

## 5 Kinetic theory viewpoint: role of "collision invariants"

Previously, we have directly identified the microscopic motion scenario as the phase-space stochastic process to be followed by an individual particle. The usage of the hydrodynamical picture implicitly refers to the collective (ensemble) "gas" picture with its corresponding Boltzmann kinetic theory. Within the Boltzmann framework, Fokker-Planck-Kramers equations are derived under various simplifying assumptions, [12, 7], and effectively play the role of oversimplified kinetic equations, with the Boltzmann collision integral being replaced by suitable differentiable expressions.

In particular, let us notice that all hitherto considered Kramers equations have the general kinetic form:

$$\left( \partial_t + \vec{u} \cdot \vec{\nabla}_{\vec{u}} + \frac{1}{m} \vec{F} \cdot \vec{\nabla}_{\vec{u}} \right) f = C(f) \quad (97)$$

where  $f = f(\vec{x}, \vec{u}, t)$  is a sought for phase-space probability density while  $C(f)$  is a substitute for the Boltzmann collision integral,  $m$  being a mass parameter.

Would we have worked with the "normal" kinetic equation in the Boltzmann form, then a standard method to produce local conservation laws is known to employ the collision invariants and heavily relies on an explicit form of  $C(f)$  which must yield:  $\int C(f) d^3u = 0$ ,  $\int \vec{u} C(f) d^3u = \vec{0}$  and in addition  $\int \vec{u}^2 C(f) d^3u = 0$ . That implies respectively mass, momentum and energy conservation laws, as a consequence of the existence of microscopically conserved additive quantities, [12, 16], c.f. also the very recent paper [33].

A concrete form of the "collision integral"  $C(f)$  while adopted to the considered before stochastic processes, can be easily deduced from the corresponding Fokker-Planck-Kramers equations. For example, the frictional dynamics involves  $C(f) = \left( q \vec{\nabla}_{\vec{u}}^2 + \beta \vec{u} \cdot \vec{\nabla}_{\vec{u}} \right) f$  while by omitting the  $\beta$ -dependent term we are left with the frictionless variant  $C(f) = q \vec{\nabla}_{\vec{u}}^2 f$ .

It is important to notice that only  $\int C(f) d^3u = 0$  holds true identically for both frictional and frictionless cases. The frictional case does not respect two other kinetic identities. On the other hand, the frictionless dynamics respects one more identity, namely  $\int \vec{u} C(f) d^3u = \vec{0}$ , but *does* violate the third one related to the energy conservation rule.

As an almost trivial remark we may consider an observation that one more step in the hierarchy of gradually more constrained kinetic problems may be completed by choosing suitable solutions of the Liouville equation, which we obtain by passing to the  $q \rightarrow 0$  limit in Eqs. (14), (37) and (83). That corresponds to replacing the stochastically forced dynamical systems by deterministic Newton laws, c.f. (13), (34), (71). Since for such solutions, the right-hand-side of Eq. (97) identically vanishes, we obviously have  $\int \vec{u}^2 C(f) d^3 u = 0$ .

For those Liouville-type probability densities, all three collision invariants do vanish, satisfying the demands put forward in Ref. [33] to support the "idea that the classical kinetics could be underlying quantum mechanics". The point is that there is manifestly *no* collisional kinetics involved in this particular case although all minimal requirements of Ref. [33] are fulfilled.

By performing  $q \rightarrow 0$  limits of previously considered joint velocity-space probability densities we pass smoothly to deterministic solutions of the respective Liouville equations in the familiar form of the positive Wigner functions (up to notation adjustments, see e.g. [27, 28]), with rather obvious hydrodynamical connotations.

As an illustrative example of this reasoning let us reproduce in detail the  $q \rightarrow 0$  limit of the frictionless dynamics of Section 2. We have (tilde is used to discriminate between  $q = 0$  and  $q \neq 0$  cases):

$$\tilde{\rho}(x, u, t) = \frac{1}{2\pi\sqrt{a^2 b^2}} \exp\left(-\frac{(u - u_{ini})^2}{2b^2} - \frac{(x - x_{ini} - tu)^2}{2a^2}\right). \quad (98)$$

Here, the normalization is preserved, since we have  $\int (\int \tilde{\rho}(x, u, t) dx) du = \int (\int \tilde{\rho}(x, u, t) du) dx = 1$ , while for the marginals  $\tilde{\rho}(x, t) = \int \tilde{\rho}(x, u, t) du$  and  $\tilde{\rho}(u, t) = \int \tilde{\rho}(x, u, t) dx$ , in the very same  $q \rightarrow 0$  limit we get:

$$\tilde{\rho}(u, t) = \left(\frac{1}{2\pi b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(u - u_{ini})^2}{2b^2}\right) \quad (99)$$

and

$$\tilde{\rho}(x, t) = \left(\frac{1}{2\pi(a^2 + b^2 t^2)}\right)^{\frac{1}{2}} \exp\left(-\frac{(x - x_{ini} - u_{ini} t)^2}{2(a^2 + b^2 t^2)}\right). \quad (100)$$

In order to find local moments  $\langle u \rangle_x = \int u \frac{\tilde{\rho}(x, u, t)}{\tilde{\rho}(x, t)} du$  and  $\langle u^2 \rangle_x = \int u^2 \frac{\tilde{\rho}(x, u, t)}{\tilde{\rho}(x, t)} du$  of the joint probability density  $\tilde{\rho}(x, u, t)$  we compute

$$\frac{\tilde{\rho}(x, u, t)}{\tilde{\rho}(x, t)} = \left(\frac{a^2 + b^2 t^2}{2\pi a^2 b^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\left(u - u_{ini} - \frac{b^2 t}{a^2 + b^2 t^2} (x - x_{ini} - u_{ini} t)\right)^2}{2 \frac{a^2 b^2}{a^2 + b^2 t^2}}\right) \quad (101)$$

Consequently, the local moments are given by

$$\langle u \rangle_x = u_{ini} + \frac{b^2 t}{a^2 + b^2 t^2} (x - x_{ini} - u_{ini} t) \quad (102)$$

$$\langle u^2 \rangle_x = \langle u \rangle_x^2 + \frac{a^2 b^2}{a^2 + b^2 t^2} \quad (103)$$

and the force-free Euler-type equation in the form (8), c.f. [11] is clearly recovered

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \nabla \right) \langle u \rangle_x = 2a^2 b^2 \nabla \left[ \frac{\Delta \tilde{\rho}(x, t)^{1/2}}{\tilde{\rho}(x, t)^{1/2}} \right] \doteq \nabla Q \quad (104)$$

where (c.f. eq. (10)) we may identify  $D^2 = a^2 b^2$  with a squared diffusion constant, [10, 11]. By setting formally  $D = \frac{\hbar}{2m}$  we recover the standard quantum mechanical "hydrodynamics", however with no intervention of any nontrivial kinetics, contrary to the hypothesis of Ref. [33].

Quite analogously, the planar motion in a uniform magnetic field gives rise to the Euler-type equation

$$[\partial_t + \langle \vec{u} \rangle_{\vec{R}} \nabla] \langle \vec{u} \rangle_{\vec{R}} = \frac{q_e}{mc} \langle \vec{u} \rangle_{\vec{R}} \times \vec{B} + 2a^2 b^2 \nabla \left[ \frac{\Delta \tilde{\rho}^{1/2}}{\tilde{\rho}^{1/2}} \right] \doteq \frac{\vec{F}}{m} + \vec{\nabla} Q \quad (105)$$

in conformity with the analogous result recovered in Eq. (69). Here,  $\vec{F}$  denotes the Lorentz force contribution. The harmonic case implies:

$$\left( \frac{\partial}{\partial t} + \langle u \rangle_x \frac{\partial}{\partial x} \right) \langle u \rangle_x = -\omega^2 x + 2a^2 b^2 \nabla \left[ \frac{\Delta \tilde{\rho}^{1/2}}{\tilde{\rho}^{1/2}} \right] \quad (106)$$

again with no genuine kinetics involved.

## 6 Concluding remarks

We have investigated in detail exemplary solvable cases of the frictionless random dynamics. All of them share a common feature to yield the local momentum conservation law in the specific form

$$[\partial_t + \langle \vec{u} \rangle_{\vec{R}} \vec{\nabla}] \langle \vec{u} \rangle_{\vec{R}} = \frac{\vec{F}}{m} + 2d(t) \vec{\nabla} \left[ \frac{\Delta w^{1/2}}{w^{1/2}} \right] \quad (107)$$

where  $\vec{F}$  denotes external force acting on the particle, and  $d = (\det C)^{\frac{1}{n}}$  where  $C$  is the covariance matrix of random variables (vectors)  $\vec{S}$  and  $\vec{R}$  (defined for each system) and  $n$  stands for the dimension of configuration space of appropriate system.

1. free particle:  $\vec{R} = x$ ,  $F \equiv 0$ ,  $n = 1$
2. charged particle in a constant magnetic field:  $\vec{R} = (x, y)$ ,  $\vec{F} = \frac{q_e}{c} \langle \vec{u} \rangle_{\vec{R}} \times \vec{B}$ ,  $n = 2$
3. harmonically bound particle:  $\vec{R} = x$ ,  $F = -m\omega^2 x$ ,  $n = 1$

In case of harmonic and magnetic confinement, we have identified parameter range regimes that allow for a positivity of the time dependent coefficient  $d(t) \doteq D^2(t)$  (in the force-free case it is positive

with no reservations), when the pressure-type contribution in Eq. (107) acquires a characteristic form of  $-\frac{\vec{\nabla} \cdot \vec{P}}{w} = +\vec{\nabla} Q$ .

By investigating the frictionless diffusion scenario we have given support to the validity of Euler-type momentum conservation laws in a carefully defined stochastic process context. In particular, that resolves problems with a consistent theoretical framework for diffusion processes which would show up Lorentz force effects in the local mean (e.g. rotational dynamics) for long times (exceeding a single period, c.f [15, 32]). However, the frictionless motion needs developing methods of control for the otherwise untamed mean energy growth (energization phenomenon).

As a byproduct of the discussion we have demonstrated that the usage of three traditional "collision invariants" representing microscopically conserved additive quantities does not produce a sufficiently general kinetic theory background for the derivation of the probabilistic counterpart of the Schrödinger picture quantum dynamics, at variance with the recent proposal of Ref. [33]. The previous discussion indicates as well that neither dissipative nor non-dissipative stochastic phase-space processes based on the white-noise kinetics are valid candidates to that end.

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## References

- [1] J. Heinrichs, Phys. Rev. **E 47**, 3007, (1993)
- [2] J. Masoliver, Phys. Rev. **A 45**, 706, (1992)
- [3] A. L. Newman, W. I. Newman, Phys. Fluids **B 3**, 4 (1991)
- [4] A. L. Newman, Geophys. Research Lett. **17**, 1061 (1990)
- [5] G. W. Ford, M. Kac, P. Mazur, J. Math. Phys. **6**, 504, (1965)
- [6] P. Ullersma, Physica **32**, 27, (1966)
- [7] J. R. Dorfman, "*An Introduction to Chaos in Nonequilibrium Statistical Mechanics*", (Cambridge University Press, Cambridge, 1999)
- [8] J. T. Lewis, L. C. Thomas, "How to make a heat bath", in: "*Functional Integration and Its Applications*", ed. A. M. Arthurs, (Oxford University Press, Oxford, 1975)
- [9] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1, (1943)

- [10] E. Nelson, "*Dynamical Theories of Brownian Motion*" (Princeton University Press, Princeton, 1967)
- [11] P. Garbaczewski, Physica **A 285**, 187, (2000)
- [12] R. L. Liboff, "*Kinetic Theory*", (Prentice Hall Int. Ed., N. J., 1990)
- [13] P. Garbaczewski, Phys. Rev. **E 59**, 1498, (1999)
- [14] Y. L. Klimontovich, "*Statistical Theory of Open Systems*", (Kluwer, Dordrecht, 1995)
- [15] R. Czopnik, P. Garbaczewski, Phys. Rev. **E 63**, 021105, (2001)
- [16] K. Huang, "*Statistical Mechanics*" (Wiley, New York, 1963)
- [17] Z. Schuss, "*Theory and Applications of Stochastic Differential Equations*", (Wiley, New York, 1980)
- [18] C. Jarzynski, Phys. Rev. Lett. **74**, 2937, (1995)
- [19] U. Weiss, "*Quantum Dissipative Dynamics*", (World Scientific, Singapore, 1999)
- [20] W. I. Newman, private communication
- [21] B. T. Geilikman, Zh. Exp. Teor. Phys. **17**, 839, (1947)
- [22] E. Guth, Adv. Chem. Phys. **15**, 363, (1969)
- [23] R. Balescu, "*Statistical Dynamics. Matter out of Equilibrium*", (Imperial College Press, London, 1997)
- [24] P. Garbaczewski, in: "*Nonlinear Fields: Classical, Random, Semiclassical*", edited by P. Garbaczewski and Z. Popowicz (World Scientific, Singapore, 1991), p. 455
- [25] A. Kolmogorov, Annals of Mathematics **35**, 116, (1934)
- [26] A.D. Wentzell, "*A Course in the Theory of Stochastic Processes*" (McGraw-Hill, NY, 1981)
- [27] T. Takabayasi, Progr. Theor. Phys. **11**, 341, (1954)
- [28] Hai-Woong Lee, Phys. Reports **259**, 148, (1995)
- [29] J.B. Taylor, Phys. Rev. Lett. **6**, 262, (1961)
- [30] B. Kurşunoğlu, Ann. Phys. (NY) **17**, 259, (1962)
- [31] E. Nelson, "*Quantum Fluctuations*" (Princeton University Press, Princeton, 1985)

- [32] P. Garbaczewski, G. Kondrat, R. Olkiewicz, Phys. Rev. E **55**, 1401 (1997)
- [33] G. Kaniadakis, Physica **A 307**, 172, (2002)